# On uniqueness of static Einstein-Maxwell-dilaton black holes

Marc Mars[*] and Walter Simon[§*]
[*]Albert Einstein Institut,
Am Mühlenberg 1, D-14476 Golm, Germany.
[§]Institut für Theoretische Physik der Univ. Wien
Boltzmanngasse 5, A-1090, Wien, Austria

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**Abstract**

We prove uniqueness of static, asymptotically flat spacetimes with non-degenerate black holes for three special cases of Einstein-Maxwell-dilaton theory: For the coupling "$\alpha = 1$" (which is the low energy limit of string theory) on the one hand, and for vanishing magnetic or vanishing electric field (but arbitrary coupling) on the other hand. Our work generalizes in a natural, but non-trivial way the uniqueness result obtained by Masood-ul-Alam who requires both $\alpha = 1$ and absence of magnetic fields, as well as relations between the mass and the charges. Moreover, we simplify Masood-ul-Alam's proof as we do not require any non-trivial extensions of Witten's positive mass theorem. We also obtain partial results on the uniqueness problem for general harmonic maps.

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# 1 Introduction.

In 1967 W. Israel proved (roughly speaking) that static, asymptotically flat vacuum spacetimes with non-degenerate, connected horizons are Schwarzschild [1]. His method is based on the integration of two "divergence identities" constructed from the norm $V$ of the static Killing vector $\vec{\xi}$ and from the induced metric $\widehat{g}$ on a slice $\Sigma$ orthogonal to it (and their derivatives). In the sequel Israel's theorem has been generalized to include certain matter fields. In particular, Israel himself also proved uniqueness in the Einstein-Maxwell case [2]. Later, it was realized that the proof of this theorem could be better understood in terms of the $SO(2,1)$-symmetry of the "potential space" (i.e. of the target space of the corresponding harmonic map) [3, 4]. Associated with that symmetry there are conserved currents and a suitable combination of them yields, upon integration, a functional relationship between $V$ and the electrostatic potential $\phi$. Using these relations one can then apply the symmetry transformations on the target space, which reduces the problem to the vacuum case. This observation leads immediately to a further generalization, namely to Einstein-Maxwell-dilaton theory with "string coupling" ($\alpha = 1$) [4]. The symmetry group of this theory is a direct product of an "electric" and a "magnetic" $SO(2,1)$-part, (and the target space is a corresponding direct sum), whence the components can be treated individually as above. This gives uniqueness for a three-parameter family of black hole solutions found by Gibbons [5]. In fact, arguments along these lines apply to the much more general case in which the target space is a symmetric space and yield uniqueness results for solutions which arise from the Schwarzschild family by applying those symmetry transformations of the respective theory which preserve asymptotic flatness [6]. (We remark, however, that the uniqueness results in [3, 4, 6] all contain some errors or gaps).

An alternative strategy for proving uniqueness consists, in essence, of performing conformal rescalings on the spatial metric $\widehat{g}$ suitable for applying the rigidity case of a positive mass theorem. The "basic version", appropriate for non-degenerate black holes in the vacuum case, was found in 1987 by Bunting and Masood-ul-Alam [7]. These authors take two copies of the region exterior to the horizon, glue them together along the bifurcation surface and rescale the metric on this compound with a suitable (positive) function of the norm of the static Killing vector which compactifies one of the ends smoothly. The resulting space is shown to be complete and with vanishing Ricci scalar and vanishing mass. Hence, the rigidity case of the positive mass theorem implies that the rescaled metric must be flat, and the rest follows from the field equations in a straightforward manner.

The main advantage of the method by Bunting and Masood-ul-Alam is that admits disconnected horizons a priori. However, as to generalizations to matter fields, they are not so straightforward to obtain along these lines. The first part of the strategy is to find candidates for conformal factors by taking functions which transform the family of spherically symmetric black hole solutions whose uniqueness is conjectured to flat space, and express these functions in terms of $V$ and the potentials. In general there are many possibilities if matter is present. However, (as follows from our Theorem 2), for harmonic maps there is in fact a unique choice of such "factor candidates" as functions on the target space provided that the latter has the same dimension as the space of spherically symmetric black hole solutions. This is the case, in particular, for Einstein-Maxwell, where these dimensions are three. (Alternatively, the magnetic or the electric field can in advance be removed by a trivial duality transformation, which reduces the dimensions to two). These "factor candidates" are direct generalizations of the vacuum quantities, and so the same procedure as before yields uniqueness of the non-extreme Reissner-Nordström solution [8, 9].

In Einstein-Maxwell-dilaton theory, spherically symmetric black holes solutions have been studied extensively (see, e.g. [10]-[13]). We first note that in this situation no duality transformation is available to remove either the electric or the magnetic field. Assuming for the moment that the latter is absent, there is just a two-parameter family of spherically symmetric black hole solutions which, consequently, cannot define a unique "factor candidate" on the three-dimensional target space. While Masood-ul-Alam did not give a single suitable conformal factor in this situation, he made remarkable observations [14] which we reformulate as follows. Firstly, for the coupling $\alpha = 1$, and assuming some relation between the mass and the charges, he found two pairs of conformal factors ${}^{\Phi}\Omega_{\pm}$ and ${}^{\Psi}\Omega_{\pm}$ such that the Ricci scalars ${}^{\Phi}\mathcal{R}$ and ${}^{\Psi}\mathcal{R}$ corresponding to the metrics ${}^{\Phi}g^{\pm} = {}^{\Phi}\Omega_{\pm}^2 V^2 \widehat{g}$ and ${}^{\Psi}g^{\pm} = {}^{\Psi}\Omega_{\pm}^2 V^2 \widehat{g}$ satisfy ${}^{\Phi}\Omega_{\pm}^2\, {}^{\Phi}\mathcal{R} + {}^{\Psi}\Omega_{\pm}^2\, {}^{\Psi}\mathcal{R} \geq 0$. Secondly, he observed that the rigidity case of Witten's positive mass theorem has a generalization which requires just the condition above (rather than non-negativity of each Ricci-scalar) to give flatness of ${}^{\Phi}g^{\pm}$ and ${}^{\Psi}g^{\pm}$ provided that the masses of these metrics also vanish. (For the general formulation of this "conformal positive mass theorem" (CPM) c.f. Simon [15]). By adapting the remaining procedure from the vacuum case, Masood-ul-Alam then obtained uniqueness of the two-parameter subfamily of the Gibbons solutions mentioned above [14].

The achievements of the present paper are threefold. Firstly, we show (in Lemma 4) that the seemingly subtle CPMs of Masood-ul-Alam and Simon have in fact trivial proofs, based on the following fact: If the Ricci scalars $R$ and $R'$ of two metrics $h$ and $h'$ related by a conformal rescaling $h' = \Omega^2 h$ satisfy $R + \Omega^2 R' \geq 0$, then the Ricci scalar $\widetilde{R}$ of the metric $\widetilde{h} = \Omega h$ is (manifestly) non-negative (by virtue of the standard formula for conformal rescalings). In particular, Masood-ul-Alam's uniqueness result can be obtained by applying this observation to $h = {}^{\Phi}g^{\pm}$ and $h' = {}^{\Psi}g^{\pm}$, and by using the rigidity case of the standard positive mass theorem for the metric $\widetilde{h} = {}^{\Phi}\Omega\ {}^{\Psi}\Omega\ V^2 \widehat{g}$.

Secondly, we extend (in Theorem 1) Masood-ul-Alam's uniqueness results in

Einstein-Maxwell-dilaton theory to the cases with non-vanishing magnetic field (still for the coupling $\alpha = 1$) on the one hand, and to arbitrary $\alpha$ but either vanishing magnetic field or vanishing electric field on the other hand (while the generic case is still open). As to the former case, it is "underdetermined" in the sense that we have a four-dimensional target space with just a three-parameter family of spherically symmetric solutions. However, as mentioned above in connection with Israel's method, this target space splits into a direct sum on which there act "electric" and "magnetic" $SO(2,1)$ groups, respectively. On each component we can now define pairs of conformal factors ${}^{\Phi}\Omega_{\pm}$ and ${}^{\Psi}\Omega_{\pm}$ in a natural manner. Thus, exploiting the group structure in this way again reduces the problem, in essence, to the Einstein-Maxwell case.

The case of arbitrary $\alpha$ but without magnetic or electric field is the more subtle one. We have now a three-dimensional target space, with invariance group $SO(2,1)\times SO(1,1)$, and a two-parameter family of spherically symmetric black hole solutions found by Gibbons and Maeda [10]. Along with the two components of the group there come again naturally two pairs of conformal factors ${}^{\Phi}\Omega_{\pm}$ and ${}^{\Psi}\Omega_{\pm}$ such that (in the case with vanishing magnetic field) the corresponding Ricci scalars satisfy ${}^{\Phi}\Omega_{\pm}^2\, {}^{\Phi}\mathcal{R} + \alpha^2\, {}^{\Psi}\Omega_{\pm}^2\, {}^{\Psi}\mathcal{R} \geq 0$. Now we use the following extension of the previous observation: If the Ricci scalars $R$ and $R'$ of two metrics $h$ and $h'$ related by a conformal rescaling $h' = \Omega^2 h$ satisfy $R + \beta\Omega^2 R' \geq 0$ for some constant $\beta$, then the Ricci scalar $\widetilde{R}$ of the metric $\widetilde{h} = \Omega^{2\beta/(1+\beta)} h$ is (manifestly) non-negative. Thus the uniqueness proof can now be completed by taking $\beta = \alpha^2$, $h = {}^{\Phi}g$, $h' = {}^{\Psi}g$ and by applying the standard positive mass theorem to $\widetilde{h} = {}^{\Phi}\Omega^{2/(1+\beta)}\ {}^{\Psi}\Omega^{2\beta/(1+\beta)} V^2 \widehat{g}$.

Thirdly, we consider general harmonic maps. We show (in Theorem 2) how "factor candidates" are uniquely fixed on a submanifold $\mathcal{V}_{BH}$ of the target space corresponding to spherically symmetric black hole solutions. Combined with a suitable positive mass theorem this result should give uniqueness proofs for symmetry groups with more complicated decompositions than the ones considered before. In particular, if the symmetry group splits into $n \geq 2$ tractable factors like $SO(2,1)$, a generalized CPM theorem involving $n$ conformal rescalings $h_i = \Omega_i^2 h$ such that $\sum_i \alpha_i^2 \Omega_i^2 R_i \geq 0$ for constants $\alpha_i$ would be suitable. However, if such a CPM theorem were true at all, there does not at least seem to be such a trivial proof as in the case $n = 2$ discussed before.

We finally recall that, in the vacuum case P. Chruściel was able to extend the uniqueness proof such that horizons with degenerate components [16] are admitted a priori, and he also obtained a certain uniqueness result for degenerate horizons in the presence of electromagnetic fields [17]. The idea is to use an alternative conformal rescaling due to Ruback [18] (which avoids compactification) and a suitably generalized positive mass theorem by Bartnik and Chruściel [19] which allows "holes". To obtain a further generalization including dilatons with the present methods would require a "conformal" version of this positivity result, which is not known.

## 2 Basic Definitions

**Definition 1** *A smooth spacetime $(\mathcal{M}, {}^4g)$ is called a static non-degenerate black hole iff the following conditions are satisfied.*

*(1.1)* $(\mathcal{M}, {}^4g)$ *admits a hypersurface orthogonal Killing vector $\vec{\xi}$ (i.e. $\xi_{[\alpha}\nabla_\beta\xi_{\gamma]} = 0$) with a non-degenerate Killing horizon $\mathcal{H}$.*

*(1.2) The horizon $\mathcal{H}$ is of bifurcate type, i.e. the closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ contains points where the Killing vector $\vec{\xi}$ vanishes.*

*(1.3)* $(\mathcal{M}, {}^4g)$ *admits an asymptotically flat hypersurface $\Sigma$ which is orthogonal to the Killing vector $\vec{\xi}$ and such that $V^2 = -\xi^\alpha\xi_\alpha \to 1$ at infinity and $\partial\Sigma \subset \overline{\mathcal{H}}$.*

**Remarks.**

1. A Killing horizon $\mathcal{H}$ is a null hypersurface where $\vec{\xi}$ is null, non-zero and tangent to $\mathcal{H}$. The surface gravity $\kappa$ of $\mathcal{H}$ is defined as $\nabla_\alpha V^2|_{\mathcal{H}} = 2\kappa\xi_\alpha|_{\mathcal{H}}$; it is necessarily constant on each connected component of $\mathcal{H}$ (see [20]) and nonzero (by definition) for non-degenerate horizons.

2. Raćz and Wald have shown that condition (1.2) is satisfied in most cases of interest in which (1.1) holds. More precisely, when the Killing vector is complete and diffeomorphic to $\mathbb{R}$ and $\mathcal{H}$ is a trivial bundle over the set of orbits $\mathcal{H}/\vec{\xi}$ of the Killing vector, then a non-degenerate horizon is of bifurcate type or else the geodesics tangent to the Killing vector $\vec{\xi}$ reach a curvature singularity for a finite value of the affine parameter [21]. Similarly, condition (1.2) is automatically satisfied in stationary, globally hyperbolic spacetimes containing no white hole region (cf. [21]; and see [22] for the precise conditions). Thus, we could replace condition (1.2) by any of these global conditions on the spacetime.

We also remark that our only global condition is contained in (1.3). By asymptotic flatness we mean the following

**Definition 2** *A spacelike hypesurface $(\Sigma, \widehat{g})$ of $(\mathcal{M}, {}^4g)$ is called asymptotically flat iff*

*(2.1) Every "end" $\Sigma^\infty$, (which is a connected component of $\overline{\Sigma} \setminus \{$a sufficiently large compact set$\}$) is diffeomorphic to $\mathbb{R}^3 \setminus B$, where $B$ is a closed ball.*

*(2.2) On $\Sigma^\infty$ the metric satisfies (in the cartesian coordinates defined by the diffeomorphism above and with $r^2 = \sum_i (x^i)^2$)*

$$\widehat{g}_{ij} - \delta_{ij} = O^2(r^{-\delta}) \text{ for some } \delta > 0. \tag{1}$$

*(A function $f(x^i)$ is said to be $O^k(r^\alpha)$, $k \in \mathbb{N}$, if $f(x^i) = O(r^\alpha)$, $\partial_j f(x^i) = O(r^{\alpha-1})$ and so on for all derivatives up to and including the $k$th ones).*

**Remarks.**

1. In the definition above, $\overline{\Sigma}$ is the topological closure of $\Sigma$, and $\widehat{g}$ is the induced metric on $\Sigma$. Notice that our definition implies, in particular, that $\overline{\Sigma}$ is complete in the metric sense.

2. Let $q$ be a fixed point of $\vec{\xi}$ on $\overline{\mathcal{H}}$ (i.e. $q \in \overline{\mathcal{H}}$ and $\vec{\xi}(q) = 0$), which exists by assumption (1.2). Then, the connected component of the set $\{p | \ \vec{\xi}(p) = 0\}$ containing $q$ is a smooth, embedded, spacelike, two-dimensional submanifold of $\mathcal{M}$ [23, 16]. Such a component is called a bifurcation surface. By assumption (1.3), any connected component $(\partial\Sigma)_\alpha$ of the topological boundary of $\Sigma$ is contained in the closure of the Killing horizon. Thus, (section 5 in [21]), $(\partial\Sigma)_\alpha$ must be a subset of one of the bifurcation surfaces of $\vec{\xi}$. Furthermore, the induced metric $\widehat{g}$ on the hypersurface $\Sigma$ can be smoothly extended to $\Sigma \cup (\partial\Sigma)_\alpha$ (see Proposition 3.3 in [16]). Hence $(\overline{\Sigma}, \widehat{g})$ is a smooth Riemannian manifold with boundary.

Next we define the concept of coupled harmonic map between manifolds. This is useful for our purposes because the Einstein field equations for many models [6] (including Einstein-Maxwell-dilaton) in stationary spacetimes can be written as such a map.

**Definition 3** *A coupled harmonic map is a $C^2$ map $\Upsilon : \Sigma \to \mathcal{V}$ between the manifolds $(\Sigma, g)$ and $(\mathcal{V}, \gamma)$, (with $g$ a positive definite metric and $\gamma$ any metric), which extremizes the Lagrangian (-density)*

$$L = \sqrt{det\, g}\left[R - \gamma_{ab}(\Upsilon(x))g^{ij}\nabla_i\Upsilon^a(x)\nabla_j\Upsilon^b(x)\right], \tag{2}$$

*upon independent variations with respect to $g_{ij}$ and $\Upsilon^a(x)$ (Here $\nabla$ is the covariant derivative and $R$ is the Ricci scalar with respect to $g$, and $\Upsilon^c(x)$ is the expression of $\Upsilon$ in local coordinates of $\mathcal{V}$).*

**Remarks.**

1. The definition above generalizes the notion of "harmonic map" which has as Lagrangian only the second term in (2), with prescribed metric $g$ and with $\Upsilon^a(x)$ as dynamical variable.

2. For Einstein's equations with a Killing vector, the domain manifold $\Sigma$ is the space of orbits, provided this space is a manifold, and $h$ is the so-called *rescaled orbit space metric*. In the static case we are dealing with, $\Sigma$ is a hypersurface orthogonal to the Killing field $\vec{\xi}$ and $g = V^2\widehat{g}$.

# 3 Einstein-Maxwell-dilaton fields

The Einstein-Maxwell-dilaton theory is defined by the following Lagrangian (-density) on $\mathcal{M}$

$$L = \sqrt{-\det {}^4g}({}^4R - 2\nabla_\alpha\tau\nabla^\alpha\tau - e^{-2\alpha\tau}F_{\alpha\beta}F^{\alpha\beta}). \tag{3}$$

Here, ${}^4g$ is a Lorentzian metric on $\mathcal{M}$, $\tau$ is a scalar field, $F_{\mu\nu}$ is a closed 2-form and $\alpha$ is a real and positive ("coupling"-) constant. Assuming that the manifold $\mathcal{M}$ is simply connected, there exists globally a vector potential $A_\mu$ such that $F_{\alpha\beta} = 2\nabla_{[\alpha}A_{\beta]}$. Taking $g_{\mu\nu}$, $\tau$ and $A_\mu$ as dynamical variables in (3), variation with respect to $A_\mu$ implies that the 2-form $e^{-2\alpha\tau} * F_{\mu\nu} = \frac{1}{2}e^{-2\alpha\tau}\epsilon_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}$ (where $\epsilon_{\mu\nu\alpha\beta}$ is the volume form corresponding to ${}^4g$) is also closed. Hence there exists a (global) vector potential $C_\mu$ such that $*F_{\mu\nu} = 2e^{2\alpha\tau}\nabla_{[\mu}C_{\nu]}$. (Alternatively, we could have taken $C_\mu$ as dynamical variable and derived the existence of $A_\mu$). Einstein-Maxwell is contained as the particular case $\tau = \text{const}$. Other important subcases are $\alpha = 1$, which arises in string theory and as the bosonic sector of $n = 4$ supergravity, and $\alpha = \sqrt{3}$ which corresponds to Kaluza-Klein theory (i.e. a Ricci-flat Lorentzian metric on a 5-dimensional manifold admitting a spacelike Killing vector with certain specific properties).

We assume that on $\mathcal{M}$ there is a timelike, hypersurface-orthogonal Killing field $\vec{\xi}$ which also leaves the scalar and electromagnetic fields invariant. In other words, the twist vector defined by $\omega_\mu = \epsilon_{\mu\nu\sigma\tau}\xi^\nu\nabla^\sigma\xi^\tau$ vanishes, and we have $\mathcal{L}_\xi\tau = \mathcal{L}_\xi F_{\mu\nu} = 0$ where $\mathcal{L}_\xi$ is the Lie derivative along $\vec{\xi}$. We further define the electric and magnetic fields by $E_\mu = F_{\mu\nu}\xi^\nu$ and $B_\mu = e^{-2\alpha\tau} * F_{\mu\nu}\xi^\nu$. Using these definitions together with $\omega_\mu = 0$ and with the Ricci identitities and the Einstein equations, we obtain

$$0 = \nabla_{[\mu}\omega_{\nu]} = \epsilon_{\mu\nu\sigma\tau}R^\sigma{}_\rho\xi^\rho\xi^\tau = 2E_{[\mu}B_{\nu]}, \tag{4}$$

and therefore either $B_\mu = 0$ or $E_\mu = aB_\mu$ for some function $a$. In the Einstein-Maxwell case ($\tau = \text{const.}$), it is easy to see that $a = \text{const.}$, but this need not hold when the dilaton field is present. Next, the Euler-Lagrange and the Killing equations imply that $\nabla_{[\mu}E_{\nu]} = 0$ and $\nabla_{[\mu}B_{\nu]} = 0$; hence there exist electric and magnetic potentials $\phi$ and $\psi$ defined (up to constants) by $E_\mu = \nabla_\mu\phi$ and by $B_\mu = \nabla_\mu\psi$. With a suitable choice of gauge (i.e. by adding gradients of suitable functions to $A_\mu$ and $C_\mu$) we can achieve that $\mathcal{L}_\xi A_\mu = \mathcal{L}_\xi C_\mu = \mathcal{L}_\xi\phi = \mathcal{L}_\xi\psi = 0$; in this gauge the scalar potentials also satisfy $\phi = A_\mu\xi^\mu$ and $\psi = C_\nu\xi^\nu$.

We now write the Einstein-Maxwell-dilaton field equations as equations on a hypersurface $(\Sigma, \widehat{g})$ orthogonal to $\vec{\xi}$. In terms of the variables introduced above they read explicitly (with $\widehat{\nabla}$ denoting the covariant derivative with respect to $\widehat{g}$),

$$\widehat{\Delta}V = V^{-1}e^{-2\alpha\tau}\widehat{\nabla}_i\phi\widehat{\nabla}^i\phi + V^{-1}e^{2\alpha\tau}\widehat{\nabla}_i\psi\widehat{\nabla}^i\psi, \tag{5}$$

$$
\begin{aligned}
\widehat{\Delta}\tau &= -V^{-1}\widehat{\nabla}_i\tau\widehat{\nabla}^i V + \alpha V^{-2}e^{-2\alpha\tau}\widehat{\nabla}_i\phi\widehat{\nabla}^i\phi - \alpha V^{-2}e^{2\alpha\tau}\widehat{\nabla}_i\psi\widehat{\nabla}^i\psi, && (6)\\
\widehat{\Delta}\phi &= V^{-1}\widehat{\nabla}_i V\widehat{\nabla}^i\phi - 2\alpha\widehat{\nabla}_i\tau\widehat{\nabla}^i\phi, && (7)\\
\widehat{\Delta}\psi &= V^{-1}\widehat{\nabla}_i V\widehat{\nabla}^i\psi + 2\alpha\widehat{\nabla}_i\tau\widehat{\nabla}^i\psi, && (8)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{R}_{ij} = {} & V^{-1}\widehat{\nabla}_i\widehat{\nabla}_j V + 2\widehat{\nabla}_i\tau\widehat{\nabla}_j\tau - \\
& -V^{-2}e^{-2\alpha\tau}(2\widehat{\nabla}_i\phi\widehat{\nabla}_j\phi - \widehat{g}_{ij}\widehat{\nabla}_k\phi\widehat{\nabla}^k\phi) - V^{-2}e^{2\alpha\tau}(2\widehat{\nabla}_i\psi\widehat{\nabla}_j\psi - \widehat{g}_{ij}\widehat{\nabla}_k\psi\widehat{\nabla}^k\psi), \quad (9)
\end{aligned}
$$

where $\widehat{R}_{ij}$ is the Ricci tensor of $\widehat{g}$. For the trace of (9) we obtain

$$
\widehat{R} = 2\widehat{\nabla}_i\tau\widehat{\nabla}^i\tau + 2V^{-2}e^{-2\alpha\tau}\widehat{\nabla}_i\phi\widehat{\nabla}^i\phi + 2V^{-2}e^{2\alpha\tau}\widehat{\nabla}_i\psi\widehat{\nabla}^i\psi. \qquad (10)
$$

We first give a lemma on the behaviour of the fields on the horizon.

**Lemma 1** *For static Einstein-Maxwell-dilaton non-degenerate black holes, there hold the following relations on the boundary $\partial\Sigma$*

$$
\widehat{\nabla}^i V\widehat{\nabla}_i\tau\Big|_{\partial\Sigma} = 0, \qquad \widehat{\nabla}^i V\widehat{\nabla}_i\phi\Big|_{\partial\Sigma} = 0, \qquad \widehat{\nabla}^i V\widehat{\nabla}_i\psi\Big|_{\partial\Sigma} = 0. \qquad (11)
$$

*Proof.* Recall that the induced metric $\widehat{g}$ on the hypersurface $\Sigma$ can be smoothly extended to $\Sigma\cup(\partial\Sigma)_\alpha$ (see Proposition 3.3 in [16]). Since $\tau$ is also smooth, it follows from (10) that $V^{-2}e^{-2\alpha\tau}\widehat{\nabla}_i\phi\widehat{\nabla}^i\phi$ and $V^{-2}e^{2\alpha\tau}\widehat{\nabla}_i\psi\widehat{\nabla}^i\psi$ have regular extensions to $\partial\Sigma$. Now the first equation in (11) follows from (6) while the remaining two equations follow from (7) and (8). □

We can bring equations (5)-(9) to the form of a coupled harmonic map between $(\Sigma, V^2\widehat{g})$ and the four-dimensional target manifold $\mathcal{V}$ defined by $(V,\tau,\phi,\psi)\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}$ endowed with the metric

$$
ds^2 = \gamma_{ab}dx^a dx^b = 2V^{-2}dV^2 + 2d\tau^2 - 2V^{-2}(e^{-2\alpha\tau}d\phi^2 + e^{2\alpha\tau}d\psi^2). \qquad (12)
$$

Our results on this model will be restricted to three special cases, namely $\psi = 0$ $\alpha = 1$ and $\phi = 0$. In each case, we parametrize the target space $\mathcal{V}$ by variables denoted by $\Phi_A$ and $\Psi_A$ (A = -1,0,1) in terms of which the isometry group of $(\mathcal{V},\gamma)$ acts lineary. The definitions of $\Phi_A$ and $\Psi_A$ are different in the three cases, but we treat these cases independently and therefore use below the same symbols for simplicity.

Thus, in terms of the auxiliary variables $\gamma_\beta = Ve^{\beta\tau}$, $\beta\in\mathbb{R}$, $\widetilde{\phi} = \sqrt{\alpha^2+1}\,\phi$ and $\widetilde{\psi} = \sqrt{\alpha^2+1}\,\psi$ we define

$\psi = 0$:

$$\begin{aligned}
\Phi_{-1} &= \frac{1}{2}[\gamma_\alpha - \gamma_\alpha^{-1}(\widetilde{\phi}^2+1)], & \Psi_{-1} &= \frac{1}{2}(\gamma_{-1/\alpha} - \gamma_{-1/\alpha}^{-1}),\\
\Phi_0 &= \gamma_\alpha^{-1}\widetilde{\phi}, & \Psi_0 &= 0,\\
\Phi_1 &= \frac{1}{2}[\gamma_\alpha - \gamma_\alpha^{-1}(\widetilde{\phi}^2-1)], & \Psi_1 &= \frac{1}{2}(\gamma_{-1/\alpha} + \gamma_{-1/\alpha}^{-1}).
\end{aligned}$$

$\alpha = 1$:

$$\begin{aligned}
\Phi_{-1} &= \frac{1}{2}[\gamma_1 - \gamma_1^{-1}(\widetilde{\phi}^2+1)], & \Psi_{-1} &= \frac{1}{2}[\gamma_{-1} - \gamma_{-1}^{-1}(\widetilde{\psi}^2+1)],\\
\Phi_0 &= \gamma_1^{-1}\widetilde{\phi}, & \Psi_0 &= \gamma_{-1}^{-1}\widetilde{\psi},\\
\Phi_1 &= \frac{1}{2}[\gamma_1 - \gamma_1^{-1}(\widetilde{\phi}^2-1)], & \Psi_1 &= \frac{1}{2}[\gamma_{-1} - \gamma_{-1}^{-1}(\widetilde{\psi}^2-1)].
\end{aligned}$$

$\phi = 0$:

$$\begin{aligned}
\Phi_{-1} &= \frac{1}{2}(\gamma_{1/\alpha} - \gamma_{1/\alpha}^{-1}), & \Psi_{-1} &= \frac{1}{2}[\gamma_{-\alpha} - \gamma_{-\alpha}^{-1}(\widetilde{\psi}^2+1)],\\
\Phi_0 &= 0, & \Psi_0 &= \gamma_{-\alpha}^{-1}\widetilde{\psi},\\
\Phi_1 &= \frac{1}{2}(\gamma_{1/\alpha} + \gamma_{1/\alpha}^{-1}), & \Psi_1 &= \frac{1}{2}[\gamma_{-\alpha} - \gamma_{-\alpha}^{-1}(\widetilde{\psi}^2-1)].
\end{aligned}$$

Capital indices are raised and lowered with the metric $\eta_{AB} = \mathrm{diag}(1,-1,-1)$. Since we define here six variables out of the four ones $V, \tau, \phi$ and $\psi$ there must be two constraints, which read $\Phi_A\Phi^A = -1 = \Psi_B\Psi^B$. We also introduce the following quantities (which are in general *not* Ricci tensors of any metric)

$$\begin{aligned}
{}^\Phi R_{ij} &= \nabla_i\Phi^A\nabla_j\Phi_A, & {}^\Psi R_{ij} &= \nabla_i\Psi^A\nabla_j\Psi_A,\\
{}^\Phi R &= g^{ij}\,{}^\Phi R_{ij}, & {}^\Psi R &= g^{ij}\,{}^\Psi R_{ij},
\end{aligned}$$

where $\nabla$ denotes the covariant derivative of $g = V^2\widehat{g}$. We now write the coupled harmonic map field equations in terms of these variables. Since here and henceforth the case $\phi = 0$ arises from the case $\psi = 0$ via the exchange $\Phi \leftrightarrow \Psi$, we only give the latter case explicitly.

$$\Delta\Phi_A = {}^\Phi R\,\Phi_A, \qquad \Delta\Psi_A = {}^\Psi R\,\Psi_A, \tag{13}$$

$$\psi = 0: \qquad R_{ij} = \frac{2}{1+\alpha^2}({}^\Phi R_{ij} + \alpha^2\;{}^\Psi R_{ij}), \tag{14}$$

$$\alpha = 1: \qquad R_{ij} = {}^\Phi R_{ij} + {}^\Psi R_{ij}. \tag{15}$$

These equations can be obtained by varying the Lagrangian (-densities)

$$\begin{aligned}
\psi &= 0: \qquad L = \sqrt{-\det g}\left(\frac{1+\alpha^2}{2}R + \frac{g^{ij}\nabla_i\Phi^A\nabla_j\Phi_A}{\Phi_A\Phi^A} + \alpha^2\frac{g^{ij}\nabla_i\Psi^A\nabla_j\Psi_A}{\Psi_A\Psi^A}\right), \\
\alpha &= 1: \qquad L = \sqrt{-\det g}\left(R + \frac{g^{ij}\nabla_i\Phi^A\nabla_j\Phi_A}{\Phi_A\Phi^A} + \frac{g^{ij}\nabla_i\Psi^A\nabla_j\Psi_A}{\Psi_A\Psi^A}\right).
\end{aligned}$$

independently with respect to $g_{ij}$, $\Phi_A$ and $\Psi_A$ and imposing (afterwards) the constraints $\Phi_A\Phi^A = -1 = \Psi_B\Psi^B$.

The Lagrangian, the constraints and the field equations are invariant under the isometry group of the target space metric (12). In terms of the variables $\Phi^A$ and $\Psi^A$ the invariance transformations take the simple form

$$\Phi'_A = {}^{\Phi}L^A{}_B\Phi^B, \qquad \Psi'_A = {}^{\Psi}L^A{}_B\Psi^B,$$

where ${}^{\Phi}L^A{}_B$ and ${}^{\Psi}L^A{}_B$ satisfy $\eta_{AB}L^{\dagger A}{}_C L^B{}_D = \eta_{CD}$, i.e. they are elements of $SO(2,1)$. Since for vanishing magnetic and electric fields we have $\Psi_0 = 0$ and $\Phi_0 = 0$, respectively, the corresponding symmetry groups are $SO(2,1) \times SO(1,1)$, while in the case $\alpha = 1$ we have the full group $SO(2,1) \times SO(2,1)$. We will also use the notation $\Phi_a = \{\Phi_{-1}, \Phi_0\}$, $\Psi_a = \{\Psi_{-1}, \Psi_0\}$, and move these indices with the metric $\eta_{ab} = \mathrm{diag}(1,-1)$.

In the case $\alpha = 1$ Gibbons has given the general (three-parameter-) family of spherically symmetric solutions [5] (which we write here in harmonic coordinates)

$$\begin{aligned}
\Phi_a &= \frac{{}^{\Phi}M_a}{\sqrt{\rho^2 - C^2}}, \qquad \Psi_a = \frac{{}^{\Psi}M_a}{\sqrt{\rho^2 - C^2}}, \\
ds^2 &= d\rho^2 + (\rho^2 - C^2)(d\theta^2 + \sin^2\theta d\phi^2),
\end{aligned} \tag{16}$$

where ${}^{\Phi}M_a$, ${}^{\Psi}M_a$ and $C > 0$ are constants satisfying ${}^{\Phi}M_a{}^{\Phi}M^a = {}^{\Psi}M_a{}^{\Psi}M^a = C^2$. (By sub- and superscripts on multipole moments we always mean indices and not exponents).

The spherically symmetric solutions in the other two cases ($\psi = 0$ and $\phi = 0$; c.f. Gibbons and Maeda [10]) are given by the two-parameter subfamilies with $\Psi_0 = 0$ and $\Phi_0 = 0$, respectively. Clearly, the bifurcate horizon is located at $\rho = C$ in all cases.

In the next section we will prove uniqueness of precisely these classes of solutions.

We now examine in detail the asymptotic structure of the fields introduced above. The complete analysis is somewhat involved, but consists in essence of assembling and adapting bits and pieces available in the literature. The whole procedure is also quite similar to the "instanton" case considered in [27].

**Lemma 2** *On an end $(\Sigma^\infty, g)$ of a static, asymptotically flat solution of (13)-(15) there is a coordinate system $x^i$ (in general different from the one of Def. 1 but still called $x^i$) and there exist constants ${}^\Phi M^a$, ${}^\Phi M^a_i$ ${}^\Psi M^a$ and ${}^\Psi M^a_i$ such that*

$$\begin{aligned} \Phi^a &= \frac{{}^\Phi M^a}{r} + \frac{{}^\Phi M^a_i x^i}{r^3} + O^\infty(\frac{1}{r^3}), && (17)\\ \Psi^a &= \frac{{}^\Psi M^a}{r} + \frac{{}^\Psi M^a_i x^i}{r^3} + O^\infty(\frac{1}{r^3}), && (18)\\ g_{ij} &= \delta_{ij} + \frac{{}^\Phi M^a \ {}^\Phi M_a + {}^\Psi M^a \ {}^\Psi M_a}{r^4}(\delta_{ij} r^2 - x^i x^j) + O^\infty(\frac{1}{r^3}). && (19)\end{aligned}$$

*Proof.* The definition of asymptotic flatness (1) implies that $\widehat{R} = O(r^{-2-\delta})$ for some $\delta > 0$ and hence (by adjusting constants suitably) we have, from (10),

$$\tau = O^1(r^{-\epsilon}), \qquad \phi = O^1(r^{-\epsilon}), \qquad \psi = O^1(r^{-\epsilon}), \qquad \text{for some} \quad \epsilon > 0.$$

Using next the full equation (9) we obtain $\widehat{\nabla}_i \widehat{\nabla}_j V = O(r^{-2-\epsilon})$. To get information on $V$ and its partial derivatives, namely

$$1 - V = O^2(r^{-\epsilon}),$$

requires an iterative procedure which we take over from Proposition 2.2 of [28] (compare also lemma 5 of [27]). Standard results on the inversion of the Laplacians in (5)-(8) (Corollary 1 of Theorem 1 in [29]) then yield

$$\tau = O^2(r^{-\epsilon}), \qquad \phi = O^2(r^{-\epsilon}), \qquad \psi = O^2(r^{-\epsilon}).$$

It is now useful to pass to the variables $g_{ij}$, $\Phi_a$ and $\Psi_a$ which have the asymptotic behaviour

$$\Phi_a = O^2(r^{-\epsilon}), \qquad \Psi_a = O^2(r^{-\epsilon}), \qquad g_{ij} = \delta_{ij} + O^2(r^{-\epsilon}) \tag{20}$$

and to introduce harmonic coordinates, which preserves these falloff properties. Then we can write (13)-(15) in the form (the subsequent step follows an idea of Kennefick and O'Murchadha [30] and has been erroneously omitted in [27])

$$g^{ij}\partial_i\partial_j \Phi_a = O(r^{-2-3\epsilon}), \quad g^{ij}\partial_i\partial_j \Psi_a = O(r^{-2-3\epsilon}), \quad g^{ij}\partial_i\partial_j g_{kl} = O(r^{-2-2\epsilon}). \tag{21}$$

Inversion of the Laplacians now yields (20) but with $2\epsilon$ instead of $\epsilon$. Iterating this procedure sufficiently many times, we can improve the falloff on the r.h. sides of (21) to $O^2(r^{-3-\beta})$ for some $\beta > 0$. Following now [31] and [32], (but keeping here harmonic coordinates for simplicity) we can write these equations as

$$\triangle \Phi_a = O(r^{-3-\beta}) \qquad \triangle \Psi_a = O(r^{-3-\beta}) \qquad \triangle g_{ij} = O(r^{-3-\beta})$$

where $\triangle$ are now the flat Laplacians. Inversion yields the monopole terms in (17) and (18), (while the monopole term is absent in (19) due to the harmonic gauge condition), and remaining terms of $O^2(r^{-1-\beta})$. Finally, the last procedure can also be iterated to give (17)-(19) as they stand. $\square$

**Remark.** As elaborated in [32], the expansion (17)-(19) can in fact be pursued to arbitrary orders to give multipole expansions of a rather simple structure.

# 4 The uniqueness proof

In the previous section we described three special cases of Einstein-Maxwell-dilaton theory whose target spaces have similar group structures. We have exposed the theory in a way which makes these structures manifest by choosing variables which linearize the group action. This allows us to perform the uniqueness proof in close analogy with the electromagnetic case [8, 9]. The analogy suggests, in particular, the following choice of conformal factors,

$$ {}^{\Phi}\Omega_{\pm} = \frac{1}{2}(\Phi_1 \pm 1), \qquad {}^{\Psi}\Omega_{\pm} = \frac{1}{2}(\Psi_1 \pm 1), $$

and the rescaled metrics are denoted by

$$ {}^{\Phi}g^{\pm}_{ij} = {}^{\Phi}\Omega^2_{\pm} g_{ij}, \qquad {}^{\Psi}g^{\pm}_{ij} = {}^{\Psi}\Omega^2_{\pm} g_{ij}. $$

We now have a key lemma on the properties of these quantities.

**Lemma 3** *Let $(\Sigma, g, \Phi_a, \Psi_a)$ be an asymptotically flat solution of (13)-(15) with a non-degenerate black hole horizon. Then*

1. *$(\overline{\Sigma}, {}^{\Phi}g^{+})$ and $(\overline{\Sigma}, {}^{\Psi}g^{+})$ are asymptotically flat Riemannian spaces with $C^2$- metrics and with vanishing mass.*
2. *$(\overline{\Sigma}, {}^{\Phi}g^{-})$ $(\overline{\Sigma}, {}^{\Psi}g^{-})$ admit one-point compactifications $\widetilde{\Sigma} = \overline{\Sigma} \cup \Gamma$ such that $(\widetilde{\Sigma}, {}^{\Phi}g^{-})$ $(\widetilde{\Sigma}, {}^{\Psi}g^{-})$ are complete Riemannian spaces with $C^2$- metrics.*
3. *The spaces $(\overline{\Sigma}, {}^{\Phi}g^{+})$ and $(\overline{\Sigma}, {}^{\Psi}g^{+})$ can be glued together with $(\overline{\Sigma}, {}^{\Phi}g^{-})$ and $(\overline{\Sigma}, {}^{\Psi}g^{+})$, respectively, to give Riemannian spaces $(\mathcal{N}, {}^{\Phi}g)$ and $(\mathcal{N}, {}^{\Psi}g)$ with $C^{1,1}-$ metrics.*

*Proof.* The proof is identical in all three cases discussed in the preceding section (the coupling constant $\alpha$ does not appear). Moreover, since the proof consists of the identical "$\Phi$"- and "$\Psi$"-parts we only give the former explicitly.

We first show that ${}^{\Phi}\Omega_{\pm} > 0$. We define the quantities ${}^{\Phi}\Xi_{\pm} = (1 \pm \Phi_0)(\Phi_1 - \Phi_{-1})^{-1} - 1$ which satisfy ${}^{\Phi}\Xi_{+}\ {}^{\Phi}\Xi_{-} = 4{}^{\Phi}\Omega_{-}(\Phi_1 - \Phi_{-1})^{-1}$ and we note that $(\Phi_1 -$

$\Phi_{-1}) > 0$ since $V > 0$ and $\theta > 0$. Moreover, by a straightforward calculation and by using (13)-(15) we find that

$$-\nabla^i[(\Phi_1 - \Phi_{-1})^2 \nabla_i {}^\Phi\Xi_\pm] = \Delta(\Phi_1 - \Phi_{-1}) = {}^\Phi R(\Phi_1 - \Phi_{-1}) = \\ = (\nabla_i {}^\Phi\Xi_+)(\nabla^{i\Phi}\Xi_-)(\Phi_1 - \Phi_{-1})^3. \qquad (22)$$

By the min.-max. principle, the quantities ${}^\Phi\Xi_\pm$ take on their extrema on the boundary, i.e. either on $\partial\Sigma$ or at infinity. Since the black hole horizon is non-degenerate, $W \equiv \widehat{\nabla}_i V \widehat{\nabla}^i V$ is non-zero at $\partial\Sigma$. Hence $\widehat{n}_i = -W^{-1/2}\widehat{\nabla}_i V$ is a unit outward normal to $\partial\Sigma$. We have $\widehat{n}^i \widehat{\nabla}_i {}^\Phi\Xi_\pm < 0$ on $\partial\Sigma$ and so ${}^\Phi\Xi_\pm$ must in particular take on their maxima at infinity where they approach zero, from (17). Hence ${}^\Phi\Xi_\pm < 0$ on $\overline{\Sigma}$. This proves the positivity of ${}^\Phi\Omega_-$, and obviously we have ${}^\Phi\Omega_+ > {}^\Phi\Omega_-$.

In order for ${}^\Phi\Omega_-$ to qualify as conformal factor for the $C^2$- compactification we have to further show that ${}^\Phi\Omega_- = Dr^{-2} + \mathrm{O}^2(r^{-3})$ for some positive constant $D$. The asymptotic behaviour (17) yields ${}^\Phi\Xi_\pm = r^{-1}({}^\Phi M^{-1} \pm {}^\Phi M^0) + O(r^{-2})$ where, due to ${}^\Phi\Xi_\pm < 0$, we have $|{}^\Phi M^0| \le |{}^\Phi M^{-1}|$. To exclude the cases ${}^\Phi M^0 = \pm {}^\Phi M^{-1}$ we conclude indirectly: if one of these relations held, (17) would give, for the corresponding $\Xi_\mp$, the expansion ${}^\Phi\Xi_\mp = r^{-3}({}^\Phi M_i^{-1} x^i \mp {}^\Phi M_i^0 x^i) + O^2(r^{-3})$. This would, however, contradict ${}^\Phi\Xi_\mp < 0$ unless ${}^\Phi M_i^{-1} x^i = \pm {}^\Phi M_i^0 x^i$ and thus ${}^\Phi\Xi_\pm = O^2(r^{-3})$. To proceed further we now write (22) on some end $\Sigma^\infty$ in the form

$$\triangle {}^\Phi\Xi_\pm = f^{ij}\partial_i\partial_j {}^\Phi\Xi_\pm + k^i \partial_i {}^\phi\Xi_\pm$$

with $\triangle$ denoting (as already in Sect. 3) the flat Laplacian, and $f^{ij}$ and $k^i$ are smooth functions with falloff $O(r^{-2})$. Inverting this Laplacian we observe that the leading term in the expansion of ${}^\Phi\Xi_\pm$ must be a homogeneous solution of order $O(r^{-3})$ which, on the other hand, must again be absent due to ${}^\Phi\Xi_\mp < 0$. By iteration, we arrive at ${}^\Phi\Xi_\mp \equiv 0$ on the end $\Sigma^\infty$ which now obviously contradicts ${}^\Phi\Xi_\mp < 0$. Therefore $4D = {}^\Phi M^{-1\Phi} M^{-1} - {}^\Phi M^{0\Phi} M^0 > 0$ as claimed. It is now clear that the derivatives of ${}^\Phi\Omega_-$ also have the required falloff.

Next, since ${}^\Phi\Omega_+ = 1 + O^2(r^{-2})$ and since $g_{ij}$ has vanishing mass, the latter is also true for ${}^\Phi g_{ij}^\pm$.

Finally, to do the matching we use again standard results (see e.g. [33]). We first show that the induced metric on $\partial\Sigma$ is the same on $(\overline{\Sigma}, {}^\Phi g^-)$ and on $(\overline{\Sigma}, {}^\Phi g^+)$. This is the case because the metrics are ${}^\Phi g^\pm = ({}^\Phi\Omega_\pm V)^2 (V^{-2} g) = ({}^\Phi\Omega_\pm V)^2 \widehat{g}$ and $\widehat{g}$ extends smoothly to $\partial\Sigma$ (see Proposition 3.3 of [16]) and $V^\Phi\Omega_\pm$ are regular at $\partial\Sigma$. Furthermore, the explicit expressions of ${}^\Phi\Omega_\pm$ show that $V^\Phi\Omega_+ = V^\Phi\Omega_-$ at $V = 0$.

The other junction condition is that the second fundamental forms of $\partial\Sigma$ with respect to the unit outward normals of $(\overline{\Sigma}, {}^\Phi g^+)$ and of $(\overline{\Sigma}, {}^\Phi g^+)$ agree apart from a sign. Under a conformal rescaling $h'_{ij} = \Omega^2 h_{ij}$, the second fundamental of a hypersurface transforms as $K'_{AB} = \Omega K_{AB} - \vec{n}(\Omega) h_{AB}$, where $\vec{n}$ is the unit normal vector with respect to $h_{ij}$ and $h_{AB}$ is the induced metric on the hypersurface. We recall that the boundary $\partial\Sigma$ is totally geodesic with respect to the metric $\widehat{g}_{ij}$ (i.e. $\widehat{K}_{AB} = 0$) [1]. A simple calculation using (11) now shows that $\widehat{n}^i \widehat{\nabla}_i ({}^\Phi\Omega_\pm V)|_{\partial\Sigma} = \mp\kappa/2$. Thus, the two second fundamental forms differ by a sign and so the glued Riemannian space $(\mathcal{N}, {}^\Phi g)$ is $C^{1,1}$. □

**Remark.** Above we had to spend some efforts on proving that $\Omega_- = Dr^{-2} + O^2(r^{-3})$ with $D > 0$. In his uniqueness proof Masood-ul-Alam just assumed this behaviour of the conformal factor by postulating a corresponding relation between the mass, the scalar and the electric charge (eq. (5) of [9]). In absence of the dilaton, i.e. in Einstein-Maxwell theory, the property $D > 0$ of the respective conformal factor is a consequence of the general mass bound $M \geq |Q|$ in terms of the charge $Q$, with $M = |Q|$ implying that the spacetime is the extreme Reissner-Nordström one. This result has been obtained as a generalization of Witten's proof [34, 35]. By the same technique, generalizations of these bounds have also been given in the Einstein-Maxwell-dilaton case in [36] (and rediscovered recently in [37]). However, these bounds (proven in the general case) are not equivalent to, and do not imply $D > 0$ (shown above in the static case). We also note that our bound $D > 0$ is different from the one claimed by Gibbons and Wells [38]. The latter is an extension of the positive mass claim by Penrose, Sorkin and Woolgar [39], which itself has so far not been established rigorously anyway.

Our final lemma is the "conformal positive mass" one, whose rigidity case will be employed later.

**Lemma 4** *Let $(\mathcal{N}, h)$ and $(\mathcal{N}, h')$ be asymptotically flat Riemannian three-manifolds with compact interior and finite mass, such that $h$ and $h'$ are $C^{1,1}$ and related via the conformal rescaling $h' = \Omega^2 h$ with a $C^{1,1}$- function $\Omega > 0$. Assume further that there exists a non-negative constant $\beta$ such that the corresponding Ricci scalars satisfy $R + \beta\Omega^2 R' \geq 0$ everywhere. Then the corresponding masses satisfy $m + \beta m' \geq 0$. Moreover, equality holds iff both $(\mathcal{N}, h)$ and $(\mathcal{N}, h')$ are flat Euclidean spaces.*

*Proof.* For the Ricci scalar $\widetilde{R}$ with respect to the metric $\widetilde{h} = \Omega^{2\beta/(1+\beta)} h$ we obtain, by standard formulas for conformal rescalings

$$(1+\beta)\widetilde{R} = \Omega^{-\frac{2\beta}{1+\beta}}(R + \beta\Omega^2 R') + 2\beta(1+\beta)^{-1}\Omega^{-2}\widetilde{\nabla}_i\Omega\widetilde{\nabla}^i\Omega.$$

By the requirements of the lemma, $\widetilde{h}$ is asymptotically flat and $\widetilde{R}$ is non-negative. Hence, by virtue of the positive mass theorem and by the relation $\widetilde{m} = (1+\beta)^{-1}(m + \beta m')$ for the masses we obtain the claimed results. □

We can now easily prove our main result.

**Theorem 1** *Let $(\mathcal{M}, {}^4g)$ be a simply connected, non-degenerate static black hole solving the Einstein-Maxwell-dilaton field equations. Then $(\mathcal{M}, {}^4g)$ must be a member of the "Gibbons-Maeda-" family of solutions ([10]).*

**Remark.** The condition that $(\mathcal{M}, {}^4g)$ is simply connected is used only to guarantee the global existence of the electric and magnetic potentials $\phi$ and $\psi$. This condition fits rather naturally to black hole spacetimes. In concrete terms, if the exterior of the black hole is assumed to be globally hyperbolic, the "topological censorship theorems" of Chruściel and Wald [40] and Galloway [41] imply simply connectedness. Thus the conclusions of the theorem hold for black holes with a globally hyperbolic domain of outer communications.

*Proof.* We introduce the Ricci scalars ${}^\Phi\mathcal{R}$ and ${}^\Psi\mathcal{R}$ with respect to ${}^\Phi g_{ij}$ and ${}^\Psi g_{ij}$ (which should not be mixed up with ${}^\Phi R$ and ${}^\Psi R$), and

$$ {}^\Phi E_i = {}^\Phi\Omega^{-1}\epsilon_{ab}\Phi^a\nabla_i\Phi^b, \qquad {}^\Psi E_i = {}^\Psi\Omega^{-1}\epsilon_{ab}\Psi^a\nabla_i\Psi^b, $$

where $\epsilon_{12} = 1 = -\epsilon_{21}$. We find that

$$ \begin{aligned} &\psi = 0 : \\ &\quad {}^\Phi\Omega^2\ {}^\Phi\mathcal{R} + \alpha^2\ {}^\Psi\Omega^2\ {}^\Psi\mathcal{R} = 2\ {}^\Phi\Omega^2\ {}^\Phi g^{ij}\ {}^\Phi E_i\ {}^\Phi E_j + 2\ {}^\Psi\Omega^2\ {}^\Psi g^{ij}\ {}^\Psi E_i\ {}^\Psi E_j, \\ &\alpha = 1 : \\ &\quad {}^\Phi\Omega^2\ {}^\Phi\mathcal{R} + {}^\Psi\Omega^2\ {}^\Psi\mathcal{R} = 2\ {}^\Phi\Omega^2\ {}^\Phi g^{ij}\ {}^\Phi E_i\ {}^\Phi E_j + 2\ {}^\Psi\Omega^2\ {}^\Psi g^{ij}\ {}^\Psi E_i\ {}^\Psi E_j. \end{aligned} $$

where the r.h. sides are manifestly non-negative. Defining now $\beta = \alpha^2$ and the metrics $h = {}^\Phi g$, $h' = {}^\Psi g$ and $\widetilde{h} = \Theta^2 g$ by

$$ \begin{aligned} &\psi = 0 : \\ &\quad \Theta^{1+\beta} = {}^\Phi\Omega\ {}^\Psi\Omega^\beta, \\ &\alpha = 1 : \\ &\quad \Theta^2 = {}^\Phi\Omega\ {}^\Psi\Omega, \end{aligned} $$

we can apply the rigidity case of Lemma 4, which yields that $(\mathcal{N}, {}^\Phi g)$, $(\mathcal{N}, {}^\Psi g)$ and $(\mathcal{N}, \widetilde{g})$ are flat. This also implies that ${}^\Phi\Omega_\pm = {}^\Psi\Omega_\pm$ and hence $\Phi_1 = \Psi_1$. Furthermore, we have ${}^\Phi E_i = {}^\Psi E_i = 0$, which yields that $a_{-1}\Phi_{-1} = a_0\Phi_0$ and $b_{-1}\Psi_{-1} = b_0\Psi_0$ for some constants $a_{-1}$, $a_0$, $b_{-1}$ and $b_0$. Hence all potentials $\Phi_A$ and $\Psi_A$ are functions of just a single variable. The following one is particularly useful

$$ \Re^2 = \frac{C^2(\Phi_1 + 1)}{4(\Phi_1 - 1)} = \frac{C^2(\Psi_1 + 1)}{4(\Psi_1 - 1)}, $$

where $C^2 = {}^\Phi M_a {}^\Phi M^a = {}^\Psi M_a {}^\Psi M^a$ is defined via the asymptotic expansions (17) and (18). In fact, the field equations (14),(15) and the flatness of $(\mathcal{N}, \widetilde{g})$ imply that $\widetilde{\nabla}_i\widetilde{\nabla}_j\Re^2 = 2\delta_{ij}$, and so $\Re$ coincides with the standard radial coordinate in $\mathbb{R}^3$ (for details, c.f. the proof of Theorem 1 in [27]). To obtain the form (16) and (17) we can introduce the harmonic coordinate $\rho = \Re + C^2/4\Re$. □

# 5 Harmonic maps

In this section we consider coupled harmonic maps in general. Our aim is to obtain information on the possible conformal factors which are suitable for proving uniqueness of spherically symmetric solutions following Bunting and Masood-ul-Alam's method.

Apart from the static Einstein-Maxwell-dilaton theory examined above, other interesting theories giving rise to harmonic maps are the following: In the stationary Einstein-Maxwell-dilaton case we have a target space with a five dimensional symmetry group for any $\alpha \neq \sqrt{3}$ [25] and an eight dimensional one for $\alpha = \sqrt{3}$ (Kaluza-Klein theory). More generally, the group structures arising e.g. in stationary Einstein-Maxwell-dilaton-axion theories and in the "symplectic gravity models" have been exposed in detail [42, 43]. In particular, variables which linearize the action of the isometry group of the target space were found. It should be emphasized that the linearizing variables we use in this paper cannot be inferred from those in the stationary case because the static Einstein-Maxwell-dilaton theory (with magnetic field) is not a so-called *consistent static truncation* (see [6] for the definition) of the stationary case.

The crucial step in the uniqueness proof of [7] is to define an appropriate conformal factor on the target space which in particular rescales the spherically symmetric solutions to flat space. By imposing just this latter property (together with some technical requirements), we will (in Theorem 2 below) fix uniquely and explicitly possible "candidates" for conformal factors on a certain subset $\mathcal{V}_{BH}$ of the target space (which in some cases coincides with the whole target space). For this purpose we obviously have to study first coupled harmonic maps where both $(\Sigma, g)$ and $\Upsilon$ are spherically symmetric. Any such map must be of the form $\Upsilon = \zeta \circ \lambda$, where $\zeta : I \subset \mathbb{R} \to \mathcal{V}$ is an affinely parametrized geodesic of $(\mathcal{V}, \gamma)$ and $\lambda : \Sigma \to \mathbb{R}$ is a spherically symmetric harmonic function on $\Sigma$ (see, e.g.[6]). Thus, spherically symmetric solutions are described by geodesics in the target space. However, not all geodesics of the target space correspond to a spherically symmetric, asymptotically flat, non-degenerate black hole solution. Let us put forward the following definition.

**Definition 4** *Let $\mathcal{V}$ be the target space of a coupled harmonic map. We define $\mathcal{V}_{BH} \subset \mathcal{V}$ as $\mathcal{V}_{BH} = \{x \in \mathcal{V} |$ there exists a spherically symmetric, asymptotically flat, non-degenerate black hole spacetime whose defining geodesic in the target space passes through $x\}$.*

**Remark.** The geodesic passing through $x \in \mathcal{V}_{BH}$ which defines the spherically symmetric black hole spacetime will be denoted by $\zeta_x(s)$. Its parametrization will be uniquely fixed (without loss of generality) by demanding $\zeta_x(0) = p$, $\zeta_x(1) = x$, where $p$ is the value of $\Upsilon$ at infinity in the domain space $\Sigma^\infty$. Notice that this condition restricts the harmonic function $\lambda$ appearing in $\Upsilon = \zeta_x \circ \lambda$ to satisfy $\lambda = 0$

at infinity in $\Sigma^\infty$. It should be remarked that the geodesic $\zeta_x$ need not be unique in general.

In the following, we shall be dealing with objects on $\Sigma$ which are the pull-backs of objects on $\mathcal{V}$ under $\Upsilon$. In order to avoid cumbersome notation we shall use the same symbol for both objects. The precise meaning should become clear from the context.

**Theorem 2** *Let $\Upsilon : \Sigma^\infty \to \mathcal{V}$ be a coupled harmonic map between $(\Sigma^\infty, g)$ and $(\mathcal{V}, \gamma)$. Assume that $(\Sigma^\infty, g)$ has vanishing mass. Let $\Omega_\pm$ be positive, $C^2$ functions $\Omega_\pm : \mathcal{V} \to \mathbb{R}$ with the following properties*

*(1) For any spherically symmetric static black hole $(\Sigma_{sph}, g_{sph})$ the metric $\Omega_\pm^2 g_{sph}$ is flat.*

*(2) $(\Sigma^\infty_{sph}, (\Omega_+)^2 g_{sph})$ is asymptotically flat and $(\Sigma^\infty_{sph}, (\Omega_-)^2 g_{sph})$ admits a one-point compactification of infinity.*

*Then, $\Omega_\pm$ must take the following form on $\mathcal{V}_{BH}$.*

$$\Omega_+(x) = \cosh^2\left(\sqrt{\frac{\dot\zeta^A(x)\dot\zeta_A(x)}{8}}\right), \quad \Omega_-(x) = \sinh^2\left(\sqrt{\frac{\dot\zeta^A(x)\dot\zeta_A(x)}{8}}\right), \quad \forall x \in \mathcal{V}_{BH}$$

*where $\vec{\dot\zeta}_x(x)$ is the tangent vector at $x$ of a geodesic $\zeta_x(s)$ in $(\mathcal{V}, \gamma)$ defining a spherically symmetric black hole $(\Sigma_{sph}, g_{sph})$.*

*Proof.* Under a conformal rescaling $g' = \Omega^2 g$, where $\Omega$ is a function $\mathcal{V} \to \mathbb{R}$, the Ricci scalar transforms as

$$\Omega^2 R' = \left(\gamma_{AB} - 4\frac{D_A D_B \Omega}{\Omega} + 2\frac{D_A \Omega D_B \Omega}{\Omega^2}\right) \nabla_i \Upsilon^A \nabla^i \Upsilon^B \tag{23}$$

where $D$ is the covariant derivative on $\mathcal{V}$. Let $x \in \mathcal{V}_{BH}$ and $\zeta_x$ be the geodesic in $(\mathcal{V}, \gamma)$ giving rise to the spherically symmetric black hole. Applying the transformation (23) to this spherically symmetric black hole and using Condition (1) we obtain, with $\Omega_\pm = (\sigma^\pm)^2$,

$$0 = \left(\gamma_{AB} - 8\frac{D_A D_B \sigma^\pm}{\sigma^\pm}\right) \dot\zeta^A(\lambda)\dot\zeta^B(\lambda)\nabla_i\lambda\nabla^i\lambda. \tag{24}$$

Defining $\tilde\sigma^\pm = \sigma^\pm \circ \zeta_x$, equation (24) becomes, after using the fact that $\zeta_x$ is a geodesic,

$$\frac{d^2\tilde\sigma^\pm(s)}{ds^2} = \frac{N(x)}{8}\tilde\sigma^\pm(s) \tag{25}$$

where $N(x) = \dot{\zeta}^A(x)\dot{\zeta}_A(x)$. Condition (2) imposes, first of all, that $\Omega_+(p) = 1$ and $\Omega_-(p) = 0$, or, equivalently,

$$\tilde{\sigma}^{\pm}(0) = 1/2 \pm 1/2. \tag{26}$$

Furthermore, under a conformal rescaling $g' = \sigma^4 g$, the mass changes according to

$$m - m' = \frac{1}{2\pi}\int_{S^\infty} \nabla_i \sigma dS^i, \tag{27}$$

where $S^\infty$ stands for the sphere at infinity in $\Sigma^\infty$. For the conformal factor $\sigma^+$, the right-hand side is zero because the metric $g_{sph}$ has vanishing mass and $(\sigma^+)^4 g_{sph}$ is flat. Similarly, for $\sigma^-$, infinity is compactified to a point and so the right-hand side must also vanish (the sphere at infinity becomes a point). Let $S_r$ be a sphere or radius $r$ in $\Sigma^\infty_{sph}$ ($r$ sufficiently large). Then (27) implies

$$0 = \lim_{r\to\infty}\int_{S_r}\left(\left.\frac{d\tilde{\sigma}^{\pm}}{ds}\right|_{s=\lambda}\nabla_i \lambda dS^i\right). \tag{28}$$

A trivial analysis of the Laplace equation for spherically symmetric functions in a spherically symmetric, asymptotically flat spacetime shows that $\nabla_i \lambda = \mathrm{O}(r^{-2})$. Thus, (28) implies $\frac{d\tilde{\sigma}^{\pm}}{ds}|_{s=0} = 0$. The unique solution of the ODE (25) fulfilling this initial condition and (26) is

$$\sigma^+(x) = \cosh\left(\sqrt{\frac{N(x)}{8}}\right), \quad \sigma^-(x) = \sinh\left(\sqrt{\frac{N(x)}{8}}\right), \quad \forall x \in \mathcal{V}_{BH}$$

and the lemma follows. $\square$.

**Remark.** In vacuum and in Einstein-Maxwell theory, $\mathcal{V}_{BH}$ coincides with the target space and hence the theorem above yields unique conformal factors (which coincide with the ones used in [8, 9]). This result is already quite remarkable. In fact, there are infinitely many possibilities to combine the potentials $V$ and $\phi$ to factors which rescale the Reissner-Nordström metric to the flat one, and which have the right boundary conditions. For instance, we can just take $\Omega$ to be either a suitable function only of $V$ or only of $\phi$. However, such conformal factors would in general depend explicitly on the mass $M$ and the charge $Q$ of the solution and therefore would not be, strictly speaking, functions on $\mathcal{V}$. This would make it very difficult to prove *a priori* that the rescaled metrics yield non-negative rescaled Ricci scalars for any solution of the coupled harmonic equations with the appropriate boundary conditions. It is precisely the assumption that the conformal factors are functions of the target space only which in general allows us to restrict them substantially.

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